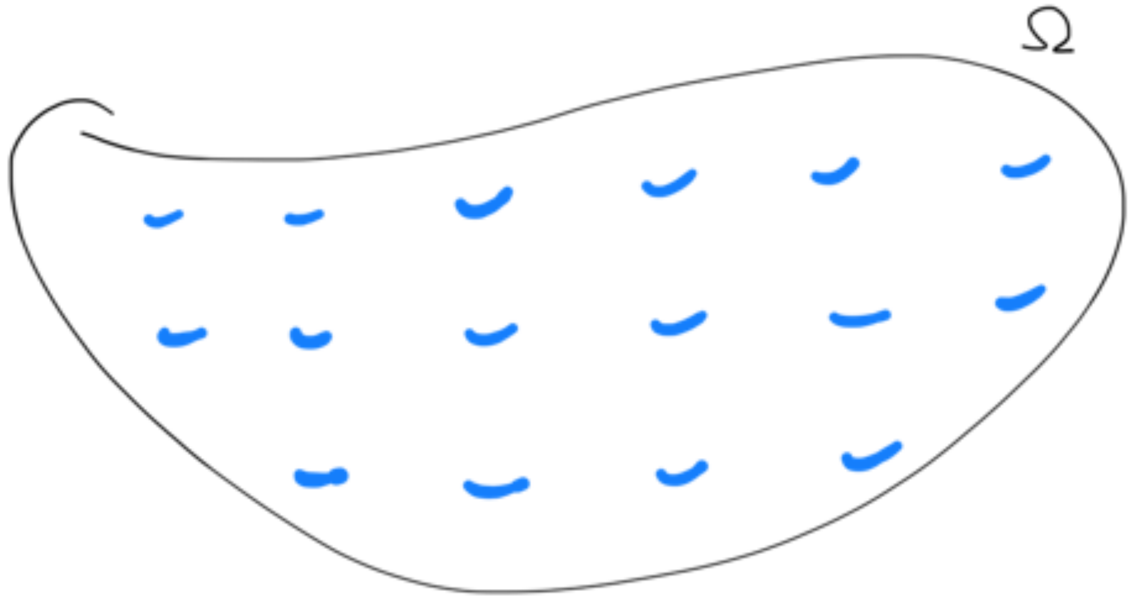
Introduction to Homogenisation

Motivation: Physical system with fine periodic structure:



e.g. wave propagation in crystal.

Simplest interesting case:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

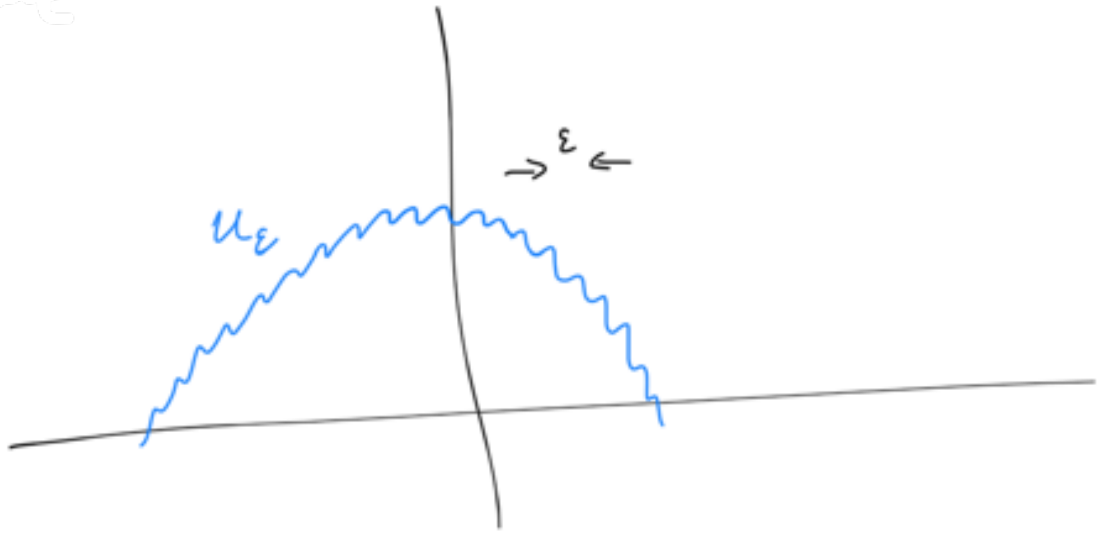
where $A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ and $A \in L^\infty(\mathbb{R}^N; \mathbb{R}^{N\times N})$ is

1-periodic and $\exists\, \alpha, \beta > 0$:

$$\begin{aligned} \alpha|\xi|^2 &\leq \xi \cdot A\xi \\ |A\xi| &\leq \beta|\xi| \end{aligned} \qquad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N.$$

Solution $u = u_\varepsilon$ expected to oscillate on length scale ε.
Bad.

$\rightsquigarrow$ Approximate u_ε by simpler function u_0 wich describes macroscopic behaviour of u_ε, but without "wiggles"



$\rightsquigarrow$ Question:

- Does (u_ε) converge for $\varepsilon \to 0$?
- If so, can limit u_0 be characterised by some reasonable PDE?

One-dimensional case

$\Omega = (a,b) \subset \mathbb{R}$.

$$\frac{d}{dx}\left(a_\varepsilon \frac{du}{dx}\right) = f \quad , \qquad a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$$

Weak formulation:

$$\int_a^b a_\varepsilon u_\varepsilon' \phi' \, dx = \int_a^b f \phi \, dx$$

$\rightsquigarrow$ a-priori bound:

$$\|u_\varepsilon\|_{H^1(a,b)} \leq C \|f\|_{L^2(a,b)}$$

$\rightsquigarrow$ subsequence $u_\varepsilon \rightharpoonup u$ in H^1.

By periodicity, $a_\varepsilon \xrightarrow{*} \langle a \rangle$ in $L^\infty(a,b)$, where

$$\langle a \rangle = \int_0^1 a(y) \, dy$$

Denote $p_\varepsilon := a_\varepsilon u_\varepsilon'$. Then

$$\left.\begin{aligned} \|p_\varepsilon\|_{L^2}^2 &\le \|a\|_{L^\infty}^2 \|u_\varepsilon'\|_{L^2}^2 \\ &\le C \|f\|_{L^2}^2 \\ \|p_\varepsilon'\|_{L^2}^2 &= \|f\|_{L^2}^2 \end{aligned}\right\} \Longrightarrow p_\varepsilon \text{ bdd. in } H^1$$

Compact embedding $\Longrightarrow$ Subsequence $p_\varepsilon \xrightarrow{\varepsilon \to 0} p$ in L^2

$\Longrightarrow \quad a_\varepsilon^{-1} p_\varepsilon \longrightarrow \langle a^{-1} \rangle p$ weakly in L^2

But also $a_\varepsilon^{-1} p_\varepsilon = u_\varepsilon' \longrightarrow u'$ weakly in L^2

$\Longrightarrow \quad \langle a^{-1} \rangle p = u'$

Use that $p' = f$:

$$f = p' = \frac{d}{dx}\left(\langle a^{-1} \rangle^{-1} \frac{du}{dx} \right)$$

"Homogenised problem".